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ASYMPTOTICALLY INVARIANT SEQUENCES AND AN ACTION OF SL(2, Z) ON THE 2-SPHERE

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ABSTRACT

Let G be a countable group which acts non-singularly and ergodically on a Lebesgue space $(X, \mathcal{S}, \mu)$. A sequence (B_n) in $\mathcal{S}$ is called *asymptotically invariant* in $\lim_n \mu(B_n \Delta gB_n) = 0$ for every $g \in G$. In this paper we show that the existence of such sequences can be characterized by certain simple assumptions on the cohomology of the action of G on X. As an explicit example we prove that a natural action of SL(2, Z) on the 2-sphere has no asymptotically invariant sequences. The last section deals with a particular cocycle for this action which has an interpretation as a random walk on the integers with "time" in SL(2, Z).

1. Introduction

Let G be a countable group which acts nonsingularly and ergodically on a standard nonatomic probability space $(X, \mathcal{S}, \mu)$. Ergodicity means, of course, that there exist no nontrivial G-invariant Borel sets in X. For some group actions, however, there exist Borel sets which are "almost" G-invariant. More specifically, one can find sequences $(B_n) \subset \mathcal{S}$ such that $\lim_n \mu(B_n \Delta gB_n) = 0$ for every $g \in G$ and (to exclude the obvious examples) such that $\lim_n \inf \mu(B_n) . (1 - \mu(B_n)) > 0$. Such sequences are called *asymptotically invariant* (a.i.), and their existence seems to be connected with a very weak form of Rokhlin's theorem. Hyperfinite group actions have therefore an abundance of a.i. sequences, a fact that has been exploited in [1]. At the other end of the spectrum one can find groups which have no such sequences, and which are therefore in particular not hyperfinite, since the collection of a.i. sequences is easily seen to be an invariant under weak equivalence. In Section 2 we give a cohomological characterization of group actions without a.i. sequences: they are precisely those group actions for which the coboundaries form a closed subgroup in the group of 1-cocycles (the precise definitions are given later in this section).

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This not only implies that the corresponding first cohomology group is "nice" in the sense that it is a polish group in its natural quotient topology, but it also removes many problems usually present when dealing with cocyles with noncompact range (or, equivalently, with noncompact skew product extensions). We refer to Proposition 2.7 and Corollary 2.8 for the details.

In Section 3 we prove that a natural action of $\mathrm{SL}(2, Z)$ on the 2-sphere (or on the 2-torus) has no a.i. sequences, and therefore enjoys all the properties discussed in Section 2. Since it is not known whether a countable group G, which acts nonsingularly and ergodically on a standard measure space, must have a nontrivial first cohomology, we prove the existence of a nontrivial 1-cocycle for $\mathrm{SL}(2, Z)$ on S^2 in Section 4. This cocycle has a simple interpretation as a (deterministic) random walk on the integers, indexed by $\mathrm{SL}(2, Z)$ on S^2 as "time". In fact, we prove that the corresponding skew product action of $\mathrm{SL}(2, Z)$ on $S^2 \times Z$ is ergodic.

In this context it is interesting to note that the existence of a.i. sequences for an action of a countable group G automatically forces the first cohomology of this action to be nontrivial. Another important observation follows from Proposition 2.7: If G has no a.i. sequences, its asymptotic ratio set (= Krieger invariant) can only take the values R, $\{\lambda^n : n \in Z\}$ and $\{0\}$, so that G cannot be of type II (cf. Remark 2.9).

We now turn to the notation and definitions used in this note. T will denote the circle group, R the reals, Z the integers. If $(X, \mathcal{S}, \mu)$ is a standard nonatomic probability space, and if A is a locally compact second countable abelian group, we write $U(X, A)$ for the group of Borel maps from X to A under pointwise addition and under the topology of convergence in measure. If G is a countable group which acts nonsingularly and ergodically on $(X, \mathcal{S}, \mu)$, $U(X, A)$ becomes a G-module in the obvious way, and we can define the first cohomolgy $H^1(G, U(X, A)) = Z^1(G, U(X, A))/B^1(G, U(X, A))$ as in [3], [7] or [8]. $Z^1(G, U(X, A))$ consists of all Borel maps $a : G \times X \to A$ with

$$(1.1) \qquad a(g_1, g_2x) - a(g_1g_2, x) + a(g_2, x) = 0 \qquad \mu\text{-a.e. } (x)$$

for every g_1, g_2 in G, and $B^1(G, U(X, A))$ is the subgroup of functions $a : G \times X \to A$ of the form

$$(1.2) \qquad a(g, x) = f(gx) - f(x),$$

where $f : X \to A$ is a Borel map. In its natural topology $Z^1(G, U(X, A))$ is a polish group (cf. [3], [8]). In order to avoid the necessity of restricting ourselves to free actions, we shall occasionally consider a closed subgroup $Z_0^1(G, U(X, A))$

of $Z^1(G, U(X, A))$ which is defined as the group of all Borel maps $a : G \times X \to A$ which satisfy (1.1), and for which

$$\mu(\{x : a(g,x) \neq 0\} \cap \{x : gx = x\}) = 0 \tag{1.3}$$

for every $g \in G$. It is clear that, for a free action of G, $Z^1(G, U(X, A)) = Z^1_0(G, U(X, A))$. We furthermore define $H^1_0(G, U(X, A))$ as $Z^1_0(G, U(X, A))/B^1(G, U(X, A))$. *To avoid trivialities we shall assume throughout this paper that* A *has at least two elements.*

Since we shall have to refer to it frequently, we shall give an explicit definition of the *essential* (*or asymptotic*) *range* $\bar{E}(a)$ of a cocycle $a : G \times X \to A$ (cf. [8, definition 3.1]). Let $\bar{A}$ denote the one-point-compactification of A if A is noncompact, and put $\bar{A} = A$ otherwise. An element $\alpha \in \bar{A}$ is said to be in $\bar{E}(a)$ if, for every neighbourhood $N(\alpha)$ of α in $\bar{A}$, and for every Borel set $B \subset X$ with $\mu(B) > 0$,

$$\mu\left(\bigcup_{g \in G} (B \cap g^{-1}B \cap \{x : a(g,x) \in N(\alpha)\})\right) > 0. \tag{1.4}$$

We put $E(a) = A \cap \bar{E}(a)$ and recall that $E(a)$ is a closed subgroup of A. Furthermore we have $\bar{E}(a) = \{0\}$ if and only if a is a coboundary. The symbol $r^*(a)$ is perhaps more widely used to denote the asymptotic range of a cocycle a, but I shall conform with the notation in [8], since we frequently have to refer to techniques presented there.

2. Asymptotically invariant sequences and cohomology

Let G be a countable group, $(X, \mathcal{S}, \mu)$ a standard nonatomic probability space, and let $(g, x) \to gx$ be a nonsingular ergodic action of G on $(X, \mathcal{S}, \mu)$. Motivated by [1], we call a sequence $(B_n) \subset \mathcal{S}$ *asymptotically invariant* if

$$\lim_n \mu(B_n \Delta g B_n) = 0 \tag{2.1}$$

for every $g \in G$. An asymptotically invariant sequence is called *trivial* if $\lim_n (\mu(B_n) . (1 - \mu(B_n)) = 0$, and *nontrivial* otherwise. For the rest of this paper, we shall abbreviate *nontrivial asymptotically invariant* as n.a.i. The following assertion is obvious.

PROPOSITION 2.1. *The existence of* n.a.i. *sequences is an invariant under weak equivalence.*

PROPOSITION 2.2. *Suppose that the action of G on $(X, \mathcal{S}, \mu)$ is hyperfinite. Then there exists a* n.a.i. *sequence $(B_n) \subset \mathcal{S}$.*

PROOF. Using Proposition 2.1 we assume that $G = Z$, and we write $(n, x) \rightarrow V^n x$ for the action of Z on $(X, \mathcal{S}, \mu)$, where V is a nonsingular ergodic automorphism. By Rokhlin's theorem there exists, for every $\varepsilon > 0$, a positive integer n and a set $A_\varepsilon \in \mathcal{S}$ with $V^k A_\varepsilon \cap V^l A_\varepsilon = \emptyset$ for $0 \leqq k < l \leqq n-1$, $\mu(\bigcup_{k=0}^{n-1} V^k A_\varepsilon) > 1 - \varepsilon$, and with $\max_{0 \leqq k \leqq n-1} \mu(V^k A_\varepsilon) < \varepsilon$. Put $m_\varepsilon = \min\{m \geqq 0 : \mu(\bigcup_{k=0}^{m} V^k A_\varepsilon) \geqq \frac{1}{2}\}$, and let $B_\varepsilon = \bigcup_{k=0}^{m_\varepsilon} V^k A$. Setting now $\varepsilon_k = 1/k$ and $C_k = B_{\varepsilon_k}$, we get

$$\frac{1}{2} \leqq \mu(C_k) < \frac{1}{2} + \frac{1}{k}$$

and

$$\mu(C_k \Delta V C_k) < \frac{2}{k}$$

for every $k \geqq 1$. Hence (C_n) is n.a.i., and the proposition is proved.

We now turn to the main concern of this section, the link between n.a.i. sequences and cohomology. Our next proposition investigates this connection in a special case.

PROPOSITION 2.3. *The following two conditions are equivalent.*

(1) *There exist no* n.a.i. *sequences for G on $(X, \mathcal{S}, \mu)$,*

(2) *$B^1(G, U(X, T))$ is a closed subgroup of $Z^1(G, U(X, T))$.*

PROOF. We shall first assume that (1) is satisfied, but not (2). Then there exists a cocycle $a : G \times X \rightarrow T$, which is not a coboundary, and a sequence (f_n) of Borel maps from X to T such that, for every $g \in G$, $\lim_n f_n \cdot g / f_n = a(g, .)$ in measure. We fix $\varepsilon > 0$ and choose points $1 = c_1, c_2, \cdots, c_p$ in T such that $|c_1 - c_p| < \varepsilon$ and $|c_i - c_{i+1}| < \varepsilon$ for $i = 1, \cdots, p-1$. Put

$$A^{(i)}_{m,n} = \{x : |f_m(x)/f_n(x) - c_i| < \varepsilon\},$$

for $1 \leqq m < n$, $1 \leqq i \leqq p$. For every fixed i, the sets $\{A^{(i)}_{m,n} : 1 \leqq m < n\}$ are asymptotically invariant as m tends to infinity. By condition (1) we have

$$\lim_{\substack{m<n \\ m \to \infty}} \mu(A^{(i)}_{m,n}) \cdot (1 - \mu(A^{(i)}_{m,n})) = 0.$$

On the other hand we know that, for every fixed pair of integers $m < n$, $\bigcup_{i=1}^{p} A^{(i)}_{m,n} = X$. Hence there exists, for every pair $m < n$, an integer $i(m, n)$ such that $1 \leqq i(m,n) \leqq p$, and $\lim_{m<n, m\to\infty} \mu(A^{(i)}_{m,n}) = 1$. We have now proved the following:

For every $\varepsilon > 0$ there exists a set of points $\{c_{m,n}, 1 \leqq m < n\} \subset T$ such that

$$\lim_{\substack{m<n \\ m\to\infty}} \mu\{x : |f_m(x) - c_{m,n} . f_n(x)| < \varepsilon\} = 1.$$

From this we conclude immediately the existence of a sequence (c_n) in T for which $(c_n . f_n)$ converges in $U(X, T)$ to a function f, say. Hence we get

$$a(g, x) = \lim_n f_n(gx)/f_n(x) = f(gx)/f(x)$$

for every $g \in G$, and for a.e. x, which is absurd. This contradiction shows that (1) implies (2).

To complete the proof we now assume that (2) is satisfied, but not (1). Let $\tilde{U}(X, T) = U(X, T)/T$, where T is identified with the set of constant T-valued functions on X. $\tilde{U}(X, T)$ is a polish group in the quotient topology. For every $f \in U(X, T)$, denote by $df \in Z^1(G, U(X, T))$ the coboundary $df(g, x) = f(gx) - f(x)$. The map $d : U(X, T) \to Z^1(G, U(X, T))$ is continuous, and defines a continuous injective group homomorphism $\tilde{d} : \tilde{U}(X, T) \to Z^1(G, U(X, T))$ in the obvious manner. By assumption, the range of $\tilde{d}$ is closed, so that $\tilde{d}$ is, in fact, a homeomorphism onto its image. Let now (B_n) be a n.a.i. sequence in $\mathcal{S}$, and put $f_n = e^{i \cdot \chi_{B_n}}$, where χ_{B_n} is the indicator function of B_n. If $\tilde{f}_n$ denotes the image of f_n in $\tilde{U}(X, T)$ under the quotient map, the condition of nontriviality for the asymptotically invariant sequence (B_n) means precisely that $(\tilde{f}_n)$ does not converge to the identity in $\tilde{U}(X, T)$. On the other hand we get $\lim_n df_n = 1$ from asymptotic invariance. This shows that $\tilde{d}$ cannot be a homeomorphism, contrary to our earlier assertion. Again we have arrived at a contradiction and conclude that (2) implies (1). The proof is complete.

Theorem 2.4. *Let A be a locally compact second countable abelian group, G a countable group, $(X, \mathcal{S}, \mu)$ a standard nonatomic probability space, and $(g, x) \to gx$ a nonsingular ergodic action of G on $(X, \mathcal{S}, \mu)$. The following conditions are equivalent.*

(1) *There exist no* n.a.i. *sequences for G on $(X, \mathcal{S}, \mu)$,*

(2) *$B^1(G, U(X, A))$ is a closed subgroup of $Z^1(G, U(X, A))$.*

Proof. We apply Proposition 2.3 to show that (1) and (2) are equivalent if $A = T$. For every character $\chi \in \hat{A}$, the map $\phi_\chi(a) = \chi . a$ from $Z^1(G, U(X, A))$ to $Z^1(G, U(X, T))$ is continuous, and it follows from [7, theorem 4.3] that

$B^1(G, U(X, A)) = \bigcap_{\chi \in \hat{A}} \phi_\chi^{-1}(B^1(G, U(X, T)))$. (1) implies that the set on the right hand side is closed as an intersection of closed sets, so that (1) implies (2). The converse is proved exactly as in Proposition 2.3.

For every cocycle $a \in Z^1(G, U(X, A))$, the set

$$B_a = \{\chi \in \hat{A} : \chi . a \in B^1(G, U(X, T))\}$$

is a Borel subgroup of $\hat{A}$, and there are some quite subtle restrictions on the kind of Borel subgroups that can occur (cf. [7, theorem 4.3] and [6]). In the absence of n.a.i. sequences, the situation is much simpler.

Corollary 2.5. *Suppose G has no* n.a.i. *sequences on $(X, \mathcal{S}, \mu)$. Then the set $B_a = \{\chi \in \hat{A} : \chi . a \in B^1(G, U(X, T))\}$ is a closed subgroup of $\hat{A}$, for every $a \in Z^1(G, U(X, T))$.*

Corollary 2.6. *Suppose G has no* n.a.i. *sequences on $(X, \mathcal{S}, \mu)$. If $a \in Z^1(G, U(X, R))$ has the property that B_a is uncountable, then a is a coboundary.*

Another consequences of the nonexistence of n.a.i. sequences can be stated in terms of the asymptotic range which a cocycle $a \in Z^1(G, U(X, A))$ can have.

Theorem 2.7. *Suppose G has no n.a.i. sequences on $(X, \mathcal{S}, \mu)$, and let A be a locally compact second countable abelian group. Then there exist no cocycles $a \in Z^1(G, U(X, A))$ with $\bar{E}(a) = \{0, \infty\}$. In other words, if $E(a) = \{0\}$, then a is a coboundary.*

Proof. Suppose there exists a (necessarily noncompact) A and a cocycle $a : G \times X \to A$ with $\bar{E}(a) = \{0, \infty\}$. From the condition $E(a) = \{0\}$ one concludes easily that a lies in fact in $Z^1_0(G, U(X, A))$. The structure theorem for such cocycles (cf. [8, theorem 7.22]) shows that there exists a standard nonatomic probability space $(Y, \mathcal{T}, \nu)$, a countable ergodic group of nonsingular automorphisms Γ of $(Y, \mathcal{T}, \nu)$, a surjective Borel map $\psi : X \to Y$, and a transient cocycle $a^* \in Z^1_0(\Gamma, U(Y, A))$ such that

$$\mu\psi^{-1} = \nu, \tag{2.2}$$

$$\psi(Gx) = \Gamma\psi(x) \qquad \text{for } \mu\text{-a.e. } x, \tag{2.3}$$

$$a(g, x) = a^*(\gamma, y) \qquad \text{whenever} \quad \psi(x) = y \quad \text{and} \quad \psi(gx) = \gamma y. \tag{2.4}$$

Since Γ has a transient cocycle in $Z^1_0(\Gamma, U(Y, A))$, the action of Γ on $(Y, \mathcal{T}, \nu)$ is hyperfinite (cf. [4, corollary 7.11]). Hence there exists a n.a.i. sequence $(B'_n) \subset \mathcal{T}$ (cf. Proposition 2.2). Put $B_n = \psi^{-1}(B'_n)$, for every $n \geqq 1$. Then it is easy to see that

(B_n) is n.a.i. for G on $(X, \mathcal{S}, \mu)$, by (2.2) and (2.3). This contradiction to our assumption shows that there exists no cocycle a with $\bar{E}(a) = \{0, \infty\}$. The second assertion, that a is a coboundary whenever $E(a) = \{0\}$, now follows from the fact that a is a coboundary if and only if $\bar{E}(a) = \{0\}$. The theorem is proved.

COROLLARY 2.8. *Suppose G has no* n.a.i. *sequences on $(X, \mathcal{S}, \mu)$. Let A be a locally compact second countable abelian group, and let $a \in Z^1(G, U(X, A))$. Then we have*

$$E(a) = B_a^{\perp}, \tag{2.5}$$

where B_a is given in Corollary 2.5 *and $B_a^{\perp} = \{\alpha \in A : \chi(\alpha) = 1$ for every $\chi \in B_a\}$ is the annihilator of B_a. Furthermore, $\bar{E}(a)$ is the closure of $E(a)$ in $\bar{A}$.*

PROOF. Let $a^* : G \times X \to A/E(a)$ be the cocycle $a^*(g, x) = a(g, x) + E(a)$. As in lemma 3.10 in [8] we conclude that $E(a^*) = \{0\}$, and Theorem 2.7 now implies that a^* is a coboundary. Using [8, proposition 3.12] we can find a coboundary $b : G \times X \to A$ such that $a(g, x) + b(g, x) \in E(a)$ for every g, x, and conclude that $B_a \supset E(a)^{\perp}$, i.e. that $E(a) \supset B_a^{\perp}$.

Conversely, if $\alpha \in E(a)$, it is obvious from the definition of $E(a)$ that $\chi(\alpha) = 1$ for every $\chi \in B_a$. Hence $E(a) \subset B_a^{\perp}$, and the proof is complete.

REMARK 2.9. Theorem 2.7 shows that no action of a countable group G on $(X, \mathcal{S}, \mu)$, which has no n.a.i. sequences, can be of type III (all we are saying is that the Radon Nikodym derivative

$$\rho(g, x) = \log \frac{d\mu g}{d\mu}(x)$$

cannot satisfy $\bar{E}(\rho) = \{0, \infty\}$). Another interesting point about n.a.i. sequences is the following: From Theorem 2.4 we know that $B^1(G, U(X, A))$ is closed (i.e. a polish group in its own right) if and only if G has no n.a.i. sequences on $(X, \mathcal{S}, \mu)$. Since $Z_0^1(G, U(X, A))$ is a polish group, we can conclude that $H_0^1(G, U(X, A)) \neq \{0\}$ whenever G admits n.a.i. sequences. In general the problem of nontriviality of $H_0^1(G, U(X, A))$ is still open, as far as I know.

The problem of deciding whether a particular *measure preserving* action of a countable group G on $(X, \mathcal{S}, \mu)$ has n.a.i. sequences can sometimes be reduced to investigating a certain property of the unitary representation $g \to U_g$ defined by

$$U_g f(x) = f(g^{-1} x) \tag{2.6}$$

for every $g \in G$, $f \in L^2(X, \mathcal{S}, \mu)$. The following proposition is obvious.

PROPOSITION 2.10. *Suppose the action of G on $(X, \mathcal{S}, \mu)$ is ergodic and measure preserving. Let $g \to V_g$ denote the restriction of the unitary representation (2.6) to the orthocomplement of the constant functions in $L^2(X, \mathcal{S}, \mu)$. If the representation $g \to V_g$ does not contain the identity representation weakly, the G has no* n.a.i. *sequences.*

REMARK 2.11. The nonexistence of n.a.i. sequences for a general non-singular ergodic action of a countable group G on $(X, \mathcal{S}, \mu)$ implies that his action is — in a certain sense — far removed from hyperfiniteness. In fact, G cannot be weakly equivalent to any ergodic action of a cartesian product $G_1 \times G_2$ on a product probability space $(X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2, \mu_1 \times \mu_2)$, where the action of $G_1 \times G_2$ is of the form $(g_1, g_2).(x_1, x_2) = (g_1 x_1, g_2 x_2)$, $g_i \in G_i$, $x_i \in X_i$, $i = 1,2$, and where G_1 acts hyperfinitely on $(X_1, \mathcal{S}_1, \mu_1)$. The latter action would admit n.a.i. sequences, and Proposition 2.1 would imply that G must also have such sequences, contrary to our assumption.

3. An action of PSL$(2, Z)$ on the 2-sphere

Let SL$(2, Z)$ denote the group of 2×2 matrices with integral entries and determinant 1, and let

$$\mathscr{C} = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be its centre. $G = \mathrm{PSL}(2, Z)$ is defined as $\mathrm{SL}(2, Z)/\mathscr{C}$ and we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \begin{matrix} + \\ - \end{matrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$$

for the quotient map. 1 will stand for the identity element in G. G is the free product of Z_2 and Z_3, where

$$Z_2 = \{1, h_1\}, \quad Z_2 = \{1, h_2, h_2^2\}, \quad h_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad h_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let now, for every $(p, q) \in Z^2$,

$$[p, q] = \{\pm (p, q)\},$$

and put $Z_*^2 = \{[p, q] : (p, q) \in Z^2\}$. G acts on Z_*^2 by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [p, q] = [ap + bq, cp + dq].$$

Let

(3.1) $D_1 = \{[p, q] \in Z^2_* : p \text{ and } q \text{ are relatively prime}\} \cup \{[1,0],[0,1]\}$,

and

(3.2) $$D_n = \{[np, nq] : [p, q] \in D_1\},$$

for every $n \geqq 0$. The following assertion is elementary.

Lemma 3.1. *The orbits of G in Z^2_* are precisely the sets D_n, $n \geqq 0$.*

We now fix $n \geqq 1$ and consider the action of G on D_n. Put

(3.3) $$S_0(n) = \{[n, 0], [0, n]\}$$

and

(3.4) $$S_1(n) = \{[n, n]\}.$$

Suppose we have defined $S_k(n) \subset D_n$ for $k = 0, \cdots, 2m+1$, where $m \geqq 0$. Put

(3.5) $$S_{2m+2}(n) = h_1 . S_{2m+1}(n)$$

and

(3.6) $$S_{2m+3}(n) = h_2^{\mathrm{T}} . S_{2m+2}(n) \cup (h_2^2)^{\mathrm{T}} . S_{2m+2}(n),$$

where M^{T} denotes the transpose of a matrix M. This inductive process yields a sequence $(S_k(n) : k \geqq 0)$ of subsets of D_n with some useful properties.

Lemma 3.2. $\bigcup_{k \geqq 0} S_k(n) = D_n$, *and* $S_k(n) \cap S_l(n) = \varnothing$ *whenever* $k \neq 1$. *Furthermore,* $\operatorname{card}(S_{2m+1}(n)) = \operatorname{card}(S_{2m+2}(n)) = 2^m$ *for every* $m \geqq 0$.

This is proved by an elementary, if slightly tedious, calculation.

We now turn to the action of G on S^2, the 2-sphere. Let $T^2 = \{(z_1, z_2) : z_i \in C, |z_i| = 1, i = 1,2\}$ denote the 2-torus, and let $\mathcal{T}$ be the Borel field and ν the normalized Lebesgue measure on T^2. Let $s : T^2 \to T^2$ be given by $s(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$, and put $\mathcal{T}^s = \{B \in \mathcal{T} : sB = B\}$. For every $(z_1, z_2) \in T^2$, let $[z_1, z_2] = \{(z_1, z_2), (\bar{z}_1, \bar{z}_2)\}$, put $X = \{[z_1, z_2] : (z_1, z_2) \in T^2\}$, and furnish X with the quotient topology. $\phi : T^2 \to X$ will denote the quotient map $\phi(z_1, z_2) = [z_1, z_2]$. As is well known and easily verified, X is homeomorphic to the 2-sphere S^2. In fact, we can choose an identification of X with S^2 which sends $[i, i]$ to the south pole, $[i, -i]$ to the north pole, and $E = \{[z, \pm 1] : z \in T\} \cup \{[\pm 1, z] : z \in T\}$ to the equator. $C_s = \{[e^{2\pi i s}, e^{2\pi i t}] : 0 < s, t < \frac{1}{2}\}$ is thus mapped to the southern hemisphere, and $C_n = \{[e^{2\pi i s}, e^{2\pi i t}] : 0 < s < \frac{1}{2} < t < 1\}$ to the northern hemisphere of S^2. Since we can choose this identification to be smooth everywhere on the complement of E, ν can be made to correspond to an absolutely continuous measure on S^2. For the rest of this paper we choose and fix such an identification and speak of X itself as the 2-sphere. The symbols C_n, C_s and E will always be interpreted as in this discussion. $\mathcal{S}$ will denote the Borel field of X and $\mu = \nu\phi^{-1}$. The obvious action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d) \tag{3.7}$$

of $\mathrm{SL}(2, Z)$ on T^2 yields an action of G on $X = S^2$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [z_1, z_2] = [z_1^a z_2^b, z_1^c z_2^d], \tag{3.8}$$

which is again measure preserving and ergodic. For every $[p, q] \in Z_*^2$, define $f_{[p,q]} \in L^2(X, \mathcal{S}, \mu)$ by

$$f_{[p,q]}([z_1, z_2]) = \begin{cases} 1 & \text{if } [p, q] = [0, 0], \\ 2^{\frac{1}{2}} \cdot \mathrm{Re}(z_1^p z_2^q) & \text{otherwise}, \end{cases} \tag{3.9}$$

and observe that $\{f_{[p,q]} : [p, q] \in Z_*^2\}$ is a complete orthonormal basis of $L^2(X, \mathcal{S}, \mu)$. If $g \to U_g$ is the representation of G on $L^2(X, \mathcal{S}, \mu)$ given by (2.6), we have

$$U_g f_{[p,q]} = f_{g^T \cdot [p,q]} \tag{3.10}$$

for every $g \in G$, $[p, q] \in Z_*^2$, where g^T is the transpose of g.

PROPOSITION 3.3. *Let $g \to V_g$ denote the restriction of the representation $g \to U_g$ in* (3.10) *to the orthocomplement of the constant functions in $L^2(X, \mathcal{S}, \mu)$. Then $g \to V_g$ does not contain the identity weakly.*

For the proof of Proposition 3.3 we need a simple lemma.

LEMMA 3.4. *For every $N \geqq 1$, and for all complex numbers $a_1, \cdots, a_N$, $b_1, \cdots, b_N$, we have*

$$\sum_{i=1}^{N} |a_i - b_i|^2 \geqq N \cdot \left(\left(\sum_{i=1}^{N} |a_i|^2/N \right)^{1/2} - \left(\sum_{i=1}^{N} |b_i|^2/N \right)^{1/2} \right)^2. \tag{3.11}$$

PROOF. (3.11) can be rewritten as

$$\sum_{i=1}^{N} |a_i - b_i|^2 \geqq \left(\left(\sum_{i=1}^{N} |a_i|^2 \right)^{1/2} - \left(\sum_{i=1}^{N} |b_i|^2 \right)^{1/2} \right)^2,$$

and this in turn follows from

$$\sum_{i=1}^{N} |a_i - b_i|^2 - \left(\left(\sum_{i=1}^{N} |a_i|^2\right)^{1/2} - \left(\sum_{i=1}^{N} |b_i|^2\right)^{1/2}\right)^2$$
$$= 2\left(\left(\sum_{i=1}^{N} |a_i|^2\right)^{1/2} . \left(\sum_{i=1}^{N} |b_i|^2\right)^{1/2} - \mathrm{Re}\left(\sum_{i=1}^{N} a_i\bar{b}_i\right)\right) \geqq 0,$$

by Schwarz's inequality.

To prove Proposition 3.3 it will be enough to show that there exists a constant $c_0 > 0$ with the following property: For every $f \in L^2(X, \mathcal{S}, \mu)$ with $U_{h_2}f = f$ and with $\int f d\mu = 0$, we have

$$\| U_{h_1}f - f \|^2 \geqq c_0 . \| f \|^2, \tag{3.12}$$

where $\| . \|$ denotes the norm in $L^2(X, \mathcal{S}, \mu)$. We therefore fix $f \in L^2$ with $\int f d\mu = 0$ and with $U_{h_2}f = f$, and write

$$f = \sum_{\substack{[p,q] \in Z^2 \\ [p,q] \neq [0,0]}} c_{[p,q]} f_{[p,q]}$$

for its expansion in the basis (3.9). From Lemma 3.2 and Lemma 3.4 we get

$$\begin{aligned} \| U_{h_1}f - f \|^2 &= \sum_{[p,q] \neq [0,0]} | c_{[p,q]} - c_{h_1 \cdot [p,q]} |^2 \\ &= 2 \sum_{n \geqq 1} \sum_{m \geqq 0} \sum_{[p,q] \in S_{2m+1}(n)} | c_{[p,q]} - c_{h_1 \cdot [p,q]} |^2 \qquad (3.13) \\ &\geqq 2 \sum_{n \geqq 1} \sum_{m \geqq 0} 2^m . (d_{m,n} - e_{m,n})^2, \end{aligned}$$

where

$$d_{m,n} = \left(\sum_{[p,q] \in S_{2m+1}(n)} 2^{-m} | c_{[p,q]} |^2 \right)^{1/2}$$

and

$$e_{m,n} = \left(\sum_{[p,q] \in S_{2m+2}(n)} 2^{-m} | c_{[p,q]} |^2 \right)^{1/2}.$$

Our assumption $U_{h_2}f = f$ now implies that, for every $m \geqq 0$, $d_{m+1,n} = e_{m,n}$. Continuing (3.13), we get

$$
\begin{aligned}
2\sum_{n\geqq 1}\sum_{m\geqq 0}2^m\cdot(d_{m,n}-e_{m,n})^2 &= \sum_{n\geqq 1}\sum_{m\geqq 0}2^{m+1}\cdot(d_{m,n}-d_{m+1,n})^2\\
&\geqq \sum_{n\geqq 1}\Bigg(\sum_{m\geqq 0}2^{m+1}d_{m,n}^2+\frac{1}{2}\sum_{m\geqq 1}2^{m+1}d_{m,n}^2\\
&\qquad -2^{1/2}\cdot\Big(\sum_{m\geqq 0}2^{m+1}d_{m,n}^2\Big)^{1/2}\cdot\Big(\sum_{m\geqq 1}2^{m+1}d_{m,n}^2\Big)^{1/2}\Bigg)
\end{aligned}\tag{3.14}
$$

from Schwarz's inequality. The right hand side of (3.14) is equal to

$$
\begin{aligned}
&\sum_{n\geqq 1}\left(\Big(\sum_{m\geqq 0}2^{m+1}d_{m,n}^2\Big)^{1/2}-2^{-1/2}\Big(\sum_{m\geqq 1}2^{m+1}d_{m,n}^2\Big)^{1/2}\right)^2\\
&\geqq(1-2^{-1/2})^2\sum_{n\geqq 1}\sum_{m\geqq 0}2^{m+1}d_{m,n}^2.
\end{aligned}\tag{3.15}
$$

Turning now to $\|f\|^2$, we have

$$
\begin{aligned}
\|f\|^2 &= \sum_{[p,q]\neq[0,0]}|c_{[p,q]}|^2\\
&= \sum_{n\geqq 1}\sum_{k\geqq 0}\sum_{[p,q]\in S_k(n)}|c_{[p,q]}|^2\\
&= \sum_{n\geqq 1}(|c_{[n,0]}|^2+|c_{[0,n]}|^2)+\sum_{m\geqq 0}\Bigg(\sum_{[p,q]\in S_{2m+1}(n)}|c_{[p,q]}|^2+\sum_{[p,q]\in S_{2m+2}(n)}|c_{[p,q]}|^2\Bigg)\\
&= \sum_{n\geqq 1}\Bigg(|c_{[n,0]}|^2+|c_{[0,n]}|^2+\sum_{m\geqq 0}2^m(d_{m,n}^2+e_{m,n}^2)\Bigg).
\end{aligned}\tag{3.16}
$$

Observing now that $U_{h_2}f=f$ implies $c_{[n,0]}=c_{[0,n]}=c_{[n,n]}=d_{0,n}$, we can continue (3.16) with

$$
\|f\|^2=\sum_{n\geqq 1}\Big(3d_{0,n}^2+3\sum_{m\geqq 1}2^{m-1}d_{m,n}^2\Big)\leqq 3\sum_{n\geqq 1}\sum_{m\geqq 0}2^m d_{m,n}^2.
$$

Combining this with (3.13)–(3.15), we get

$$
\|U_{h_1}f-f\|^2\geqq\frac{1}{3}\cdot(2^{1/2}-1)^2\cdot\|f\|^2,
$$

which proves (3.12). Proposition 3.3 is proved completely.

Proposition 2.10 now yields

THEOREM 3.5. *The action* (3.8) *of* PSL(2, Z) *on the 2-sphere* $(X,\mathcal{S},\mu)$ *has no* n.a.i. *sequences.*

COROLLARY 3.6. *The action* (3.7) *of* $\mathrm{SL}(2, Z)$ *on the* 2*-torus* $(T^2, \mathcal{T}, \nu)$ *has no* n.a.i. *sequences.*

PROOF. If $\mathrm{SL}(2, Z)$ has a n.a.i. sequence $(B_n) \subset \mathcal{T}$ we have in particular

$$\lim_n \nu\left(B_n \Delta \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . B_n\right) = 0.$$

Hence (C_n), with

$$C_n = B_n \cup \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . B_n,$$

will again be n.a.i. But (C_n) lies in $\mathcal{T}^s$, and is thus mapped to a n.a.i. sequence for $\mathrm{PSL}(2, Z)$ on $(X, \mathcal{S}, \mu)$. This contradiction proves the corollary.

4. A deterministic random walk on Z arising from the action of $\mathrm{PSL}(2, Z)$ on the 2-sphere

From Theorem 3.5 and from the results of Section 2 it is clear that the cohomology of $G = \mathrm{PSL}(2, Z)$ on the 2-sphere $(X, \mathcal{S}, \mu)$ must have some remarkable properties, provided that it is nonzero. In the following we shall define a cocycle $a \in Z^1(G, U(X, Z))$ with $E(a) = Z$. From the existence of such a cocycle it is easy to conclude that $H^1(G, U(X, A)) \neq 0$ for every locally compact second countable abelian group A. Since G is the free group generated by h_1 and h_2, every cocycle a is uniquely determined by the two functions $a(h_1, .)$ and $a(h_2, .)$. Moreover, we can change a by a coboundary to achieve $a(h_2, .) = 0$. Conversely, if f is any Borel map on X with

$$f . h_1 = -f, \tag{4.1}$$

there exists a unique cocycle a for G with

$$a(h_1, .) = f \tag{4.2}$$

and

$$a(h_2, .) = 0. \tag{4.3}$$

We now reaffirm our choice $A = Z$ and define $f : X \to Z$ by

$$f(x) = \begin{cases} 1 & \text{on } C_n, \\ 0 & \text{on } E, \\ -1 & \text{on } C_s. \end{cases} \tag{4.4}$$

For the definition of C_h, E and C_s we refer to the discussion following Lemma 3.2. For the rest of this section, we fix $a_0 \in Z^1(G, U(X, Z))$ to be the cocycle satisfying (4.2) and (4.3) with f given by (4.4).

The connection between cocycles and random walks is well known, and a_0 has a particularly nice interpretation as a (deterministic) random walk on Z with "time" G. Let $g \in G$, $g \neq e$, be a fixed element. g is of the form

$$g = h_1^{\beta} . h_2^{n_k} . h_1 . h_2^{n_{k-1}} . h_1 . \cdots . h_1 . h_2^{n_1} . h_1^{\alpha}, \tag{4.5}$$

where α, $\beta \in \{0, 1\}$, $n_i \in \{1, 2\}$, and $k \geqq 0$. (4.5) is unique if we insist that $\beta = 0$ whenever $k = 0$. The value $a_0(g, x)$ is obtained as follows (for the sake of simplicity we restrict ourselves to the case where $\alpha = \beta = 1$):

$$a_0(g, x) = f(x) + f(h_2^{n_1} h_1 x) + \cdots + f(h_2^{n_k} h_1 \cdots h_2^{n_1} h_1 x).$$

Put $x_0 = x$, $x_1 = h_2^{n_1} h_1 x, \cdots, x_k = h_2^{n_k} h_1 \cdots h_1 x$. $a_0(g, x)$ is then the number of these points in the northern hemisphere minus the number in the southern hemisphere. Alternatively we can describe $a_0(g, x)$ as the position reached after k steps of a random walk starting at zero, where the i-th step is ± 1, depending on whether x_i lies in the northern or the southern half of the space. If we vary the starting point of this random walk, we are led to the skew product action of G on $X \times Z$ given by

$$T_g(x, m) = (gx, m + a_0(g, x)). \tag{4.6}$$

Our next result implies in particular that this action is ergodic (cf. [8, corollary 5.4].

PROPOSITION 4.1. $E(a_0) = Z$.

PROOF. For every $g \in G$, we obtain a cocycle $a_g : Z \times X \to Z$ for the Z-action $(n, x) \to g^n x$ on $(X, \mathcal{S}, \mu)$. It is clear that, for every $g \in G$, $E(a_g) \subset E(a_0)$, and it will thus be enough to find a $g_0 \in G$ with $E(a_{g_0}) = Z$. We put

$$g_0 = h_2 h_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

and get $a_{g_0}(1, x) = a(g_0, x) = f(x)$. Recalling the notation in the discussion following Lemma 3.2 we use the map $\phi : T^2 \to X$ to pull back g_0 and a_{g_0} to T^2: Put, for every $(z_1, z_2) \in T^2$,

$$V(z_1, z_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} (z_1, z_2) = (z_1, \bar{z}_1 z_2),$$

and $\eta(z_1, z_2) = f([z_1, z_2])$. Then $\phi V = g_0\phi$, and $\eta = f \circ \phi$. Let $c: Z \times T^2 \to Z$ be the cocycle for the Z-action $(n, (z_1, z_2)) = V^n(z_1, z_2)$ on T^2 given by $c(1, (z_1, z_2)) = \eta(z_1, z_2)$. Since we obviously have $E(c) \subset E(a_{g_0})$, it will be enough to prove that $E(c) = Z$. We write T^2 additively as $\{(s, t): 0 \leqq s,\ t < 1\}$ with addition (mod 1). V is then represented as $V(s, t) = (s, t - s)$ (mod 1), and

$$\eta(s, t) = \begin{cases} 2\chi_{(0,1/2)}(t) - 1 & \text{for } 0 < s < 1/2, \\ 1 - 2\chi_{(0,1/2)}(t) & \text{for } 1/2 < s < 1, \\ 0 & \text{otherwise.} \end{cases}$$

On the V-invariant set $T_s = \{(s, t): 0 \leqq t < 1\}$, V acts as translation by $-s$, which we shall denote by R_s. Put $c_s(n, t) = c(n, (s, t))$ and recall that, for every irrational $s \in [0, 1)$, the cocycle $c_s: Z \times T_s \to Z$ for the ergodic automorphism R_s of T_s satisfies $E(c_s) = Z$ (cf. [2], or [8, theorem 12.8]). Remembering now the definition of $E(c_s)$ we get, for a.e. $s \in [0, 1)$ and for every Borel set $B \subset [0, 1)$ of positive Lebesgue measure λ,

$$\lambda\Big(\bigcup_{n \in Z} (B \cap R_s^{-n}B \cap \{t : c_s(n, t) = 1\})\Big) > 0.$$

One concludes easily that, for every Borel set $B \subset T^2$ with $\nu(B) > 0$,

$$\nu\Big(\bigcup_{n \in Z} (B \cap V^{-n}B \cap \{(s, t): c_n(s, t)) = 1\})\Big) > 0,$$

i.e. that $1 \in E(c)$. Since $E(c)$ is a subgroup of Z, we get $E(c) = E(a_0) = Z$, and the proposition is proved.

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